JUL 15 1959 RECEIVED

For presentation at ASME Annual Meeting, Nov. 29-Dec. 4, 1959, Atlantic City and for publication in ASME Journal of Heat Transfer.

APPLICATION OF VARIATIONAL METHODS TO RADIATION HEAT TRANSFER CALCULATIONS

By E. M. Sparrow
NASA, Lewis Research Center, Cleveland, Ohio

N65-83248
Code 1 Xerox
NASA TM 56253

E-519

ABSTRACT

A variational method is presented for solving a class of integral equations which arise in radiation heat transfer problems. First, to demonstrate the formulation of radiation problems in terms of integral equations, consideration is given to a system consisting of two nonblack, finite parallel plates. After a general description of the variational method, its use is illustrated by application to the parallel plate system. Comparisons are made which show very good agreement with exact solutions.

NOMENCLATURE

A	surface area
$a_1, a_2, \cdots$	constants which depend only on γ
$B(x), B(y)$	combined radiant flux (emitted and reflected) leaving a position x or y per unit time and unit area
$C_1, C_2, \cdots$	coefficients in eq. (16)
$D_1, D_2, \cdots$	coefficients in eq. (21)
F	angle factor
f	function of X
$G(X)$	arbitrary function of X
$H(x)$	radiant energy arriving at x per unit time and area
h	spacing between plates
I	variational expression, eq. (13)
K	kernel of integral equation
L	plate length
Q	over-all net heat transfer rate
q	local net heat transfer rate per unit area

Copy No. 1
W
L
E
A
H

COPY

DIVISION OF RESEARCH INFORMATION
NATIONAL AERONAUTICS
AND SPACE ADMINISTRATION
Washington 25, D. C.

Myron C. Nagumey
NASA Evaluator

TMX 56253

T	absolute temperature
X	dimensionless coordinate, x/L
x	coordinate measuring distance along lower plate from center plane
Y	dimensionless coordinate, y/L
y	coordinate measuring distance along upper plate from center plane
β	dimensionless combined flux, $B/\epsilon\sigma T^4$
γ	gap spacing ratio, h/L
ϵ	emissivity, $1 - \rho$
λ	constant parameter in eq. (12)
ρ	reflectivity, $1 - \epsilon$
σ	Boltzmann's constant
φ	angle defined on figure 1

INTRODUCTION

Following a long pericd of relatively moderate research activity, an upsurge of interest in radiation heat transfer has recently taken place. This renewed importance of radiation arises from its application to space technology. Not only does radiation provide a workable means for transferring waste heat from space vechicles, but also, it will serve as a dominant heat transfer mechanism in power plants of the future.

Between radiation heat transfer and the alternate mechanisms of conduction and convection, there exist significant differences in physical concepts and in mathematical representation. Physically, radiant energy is transferred by electromagnetic waves; while conduction and convection invclve contact between particles of matter. Mathematically, radiation problems must often be formulated as integral equations, while conduction and convection are most frequently formulated as differential equations. Of course, the basic principle of conservation of energy interrelates the energy transfer due to these three mechanisms.

The solution of integral equations is generally a formidable task even for those who are familiar with them. In view of this, it is of first importance to find approximate, but accurate techniques which will ease the computational effort in radiation problems involving integral equations. The great utility of variational methods as a tool for solving differential equations in many areas of applied science has already been demonstrated. It is the aim of this paper to present and apply a variational procedure which will be useful in solving a class of integral equations which arise in radiation heat transfer problems.

Since integral equations may be somewhat unfamiliar to the general heat transfer reader, we will first consider, by way of illustration, a relatively simple radiation problem and its governing integral equation. Then, a general description of the variational method for solving integral equations will be given. Finally, to demonstrate the way in which the variational procedure is used, we return to the illustrative radiation problem of the first section of the report. Comparison with exact solutions will provide a measure of the accuracy of the variational method.

A RADIATION PROBLEM AND ITS INTEGRAL EQUATION

Description of the problem. - Since our purpose here is to demonstrate the manner in which integral equations may arise in heat radiation problems, it is reasonable to select a simple physical situation. In this way, we retain the essential features of the formulation while avoiding the distractions of complex geometrical relationships. Consideration is given to the parallel plate system pictured on figure 1. Each plate is of length L and extends indefinitely in the direction normal to the plane of the figure. The surfaces of both plates radiate and reflect in a diffuse manner, and they possess identical graybody emissivity values ($\epsilon = \alpha = 1 - \rho$). The space between the plates may contain a nonparticipating gas or else, a vacuum. The thermal boundary conditions, chosen on the basis of simplicity, are that both plate surfaces are maintained at the same uniform temperature, and that negligible radiant energy passes inward from the external environment into the gap between the

plates.* It is clear that radiant energy will pass out the ends of the gap to the environment, and this will constitute the net heat transfer from the system.

For this and related systems, there are several results which may be of interest. First, it may be desired to know the distribution of the net heat flux along the surface of the plate, since it is this energy which must be locally supplied to maintain the prescribed plate temperature. Also of importance is the over-all heat transfer from the system. On the other hand, if various positions on the surface of a plate are being viewed by a pyrometer, then it is necessary to know the distribution of the combined (emitted and reflected) radiant flux. All of these results are furnished by the formulation that follows.

Mathematical formulation. - Now, turning to the analysis, attention is directed to a typical position x on the lower plate (see fig. 1). The combined radiation flux per unit time and unit area leaving an elementary area dA_x at x is designated as $B(x)$. This flux is composed of two parts, a direct emission and a reflection. The rate of emitted energy per unit area is given by the well-known relation: $\epsilon\sigma T^4$. To describe the reflected portion, we denote** the radiant flux arriving on x per unit time and unit area by $H(x)$; then, the amount $\rho H(x)$ is reflected. So, with this, we can write

$$B(x) = \epsilon\sigma T^4 + \rho H(x) \tag{1}$$

As it stands equation (1) contains two unknowns, B and H. However, it is easy to see that the energy arriving at x must be related to the energy which leaves positions on the upper plate. Consider a typical point y on the upper plate. The energy leaving an elementary area dA_y is

$$B(y) \cdot dA_y$$

Of this, an amount

$$B(y) \cdot dA_y \cdot dF_{y-x} \tag{2a}$$

*The extension to nonuniform (but identical) plate temperatures and to a radiating environment can easily be made without adding anything basically new to the formulation.

**The B,H notation has been taken from ref. 1, pp. 409-411.

arrives at position x on the lower plate, where dF_{y-x} is the angle factor of dA_x as viewed from y. Using the reciprocity relation

$$dF_{y-x} \cdot dA_y = dF_{x-y} \cdot dA_x$$

equation (2a) becomes

$$B(y) \cdot dA_x \cdot dF_{x-y} \tag{2b}$$

From this, it follows that the energy impinging per unit area at x due to radiation leaving y is

$$B(y) \cdot dF_{x-y} \tag{3}$$

But, x receives energy from all positions y on the upper surface, and the total contribution is found by integrating equation (3). So,

$$H(x) = \int_{y=-L/2}^{y=L/2} B(y) \cdot dF_{x-y} \tag{4a}$$

Using reference 2, pp. 19-21, it follows that the angle factor dF_{x-y} is given by the remarkably simple expression

$$dF_{x-y} = \frac{1}{2}\, d(\sin \varphi)$$

From figure 2, it is easy to see that

$$\sin \varphi = \frac{y - x}{\sqrt{(y - x)^2 + r^2}}$$

and with this, equation (4a) becomes

$$H(x) = \frac{h^2}{2} \int_{-L/2}^{L/2} B(y)\, \frac{1}{[(y - x)^2 + r^2]^{3/2}}\, dy \tag{4b}$$

Finally, the governing equation for B is obtained by introducing this expression for H into equation (1), giving

$$B(x) = \epsilon\sigma T^4 + \frac{\rho h^2}{2} \int_{-L/2}^{L/2} B(y)\, \frac{1}{[(y - x)^2 + r^2]^{3/2}}\, dy \tag{5}$$

or, in dimensionless form

$$\beta(X) = 1 + \frac{\rho r^2}{2} \int_{-1/2}^{1/2} \beta(Y)\, \frac{1}{[(Y - X)^2 + r^2]^{3/2}}\, dY \tag{6}$$

where γ now represents the aspect ratio h/L. Because of the symmetry of the problem, it is clear that the distribution of B (or of β) will be the same along the upper plate as along the lower plate. So, $\beta(X)$ and $\beta(Y)$ are the same functions, the only difference being that X and Y are interchanged as independent variables.

Equation (6) is called an intergral equation since the unknown function β appears under the integral sign as well as in other parts of the equation.* The factor

$$\frac{1}{[(Y - X)^2 + \gamma^2]^{3/2}}$$

is called the kernel of the integral equation and is usually identified by the notation $K(Y,X)$. In the particular case under consideration here, the kernel is symmetric since Y and X can be interchanged without any effect whatsoever. It is interesting to note that for an integral equation, there are no boundary conditions as there are in differential equations.

Once the integral equation has been solved, all quantities of interest immediately become available. The solution itself directly furnishes the distribution of the combined radiant flux B, which is useful in pyrometry. The net heat flux q at any position is the difference between the leaving and arriving radiation, that is

$$q = B - H \tag{7}$$

Eliminating H by use of equation (1) and noting that $1 - \rho = \epsilon$, the expression for q becomes

$$\frac{q}{\epsilon\sigma T^4} = \frac{1 - \beta\epsilon}{\rho} \tag{7a}$$

So, once the variation of β with X is known, the variation of q is easily computed. Finally, the over-all heat transfer Q (per unit width) is obtained by integrating equation (7a), giving the result

$$\frac{Q}{\epsilon\sigma T^4} = \frac{1 - \epsilon \int_{-1/2}^{1/2} \beta \cdot dX}{\rho} \tag{8}$$

*Mathematically speaking, eq. (6) is classified as a Fredholm equation of the second kind.

It is clearly seen that a knowledge of B (or β) is the key to the problem.

Before leaving this illustrative situation, it is well to make mention of the way this problem would have been treated using standard textbook methods (e.g., ref. 1, pp. 409-411). There, the gross assumption would have been made that the impinging energy H is the same for all points x. With this, it follows from equation (1) that B is also independent of x. To replace our equation (4a), the simple theory would substitute

$$H = B \cdot F_{=} \tag{9}$$

where $F_{=}$ is the angle factor for two finite parallel plates (see eq. (31-49), ref. 2). The combination of equations (9) and (1) gives

$$\frac{B}{\epsilon\sigma T^4} = \frac{1}{1 - \rho F_{=}} \tag{10}$$

and

$$q = \frac{Q}{L} = \frac{1 - F_{=}}{1 - \rho F_{=}} \sigma T^4 \tag{11}$$

where the heat transfer has been evaluated from equation (7a). The approximate formulation cannot give the variations of q or of B with x. It is only able to supply some sort of average value, whose accuracy depends on the complexity of the configuration. For the illustrative situation considered here, it is expected that the simple formulation would predict reasonably good Q values when $h \geq L$; with the accuracy decreasing as h/L decreases. For complex geometries with re-entrant corners, the simple formulation would be expected to lead to serious errors.

However, it is not our purpose here to evaluate the relative merits of the approximate versus the more exact integral formulation. It is clear that there are situations where both local information and more exact values of over-all results and needed, and in these, integral equations will be involved.

We now turn to the description of a variational method for solving a class of integral equations which arise in radiation problems.

THE VARIATIONAL PROCEDURE

Suppose that a physical problem is governed by the following integral equation

$$\beta(X) = G(X) + \lambda \int_a^b \beta(Y)K(Y,X)\, dX \tag{12}$$

where G is a given function of X, λ a constant parameter, and $K(Y,X)$ a symmetrical kernel. Now, according to the calculus of variations (ref. 3, p. 494), there corresponds to this integral equation the following variational expression I, which has a very special property

$$I = \lambda \int_a^b \int_a^b K(Y,X)\beta(X)\beta(Y)\, dXdY - \int_a^b [\beta(X)]^2\, dX + 2\int_a^b \beta(X)G(X)\, dX \tag{13}$$

Namely, that the value of I is an extremum (a maximum for our radiation problem) when it is evaluated using that function which satisfies the integral equation (12). This characteristic suggests a procedure for obtaining approximate solutions of the integral equation by using the variational expression I. The simplest scheme would be to select several trial functions for β and introduce them successively into equation (13). That function which gives the maximum value for I would be regarded as the closest approximation to the correct solution. A more systematic and satisfactory procedure for finding an extremum of the variational expression is the Ritz method. According to this approach, a set of functions $f_1(X), f_2(X), \cdots$ is selected, and with these, β is written as the following sum

$$\beta(X) = C_1 f_1(X) + C_2 f_2(X) + \cdots = \sum_{n=1}^{r} C_n f_n(X) \tag{14a}$$

$$\beta(Y) = C_1 f_1(Y) + C_2 f_2(Y) + \cdots = \sum_{n=1}^{r} C_n f_n(Y) \tag{14b}$$

where $C_1, C_2, \cdots$ are constants which remain to be determined. With this choice for β, the integrations indicated in equation (13) can be carried out. The result is an expression for I containing the $C_1, C_2, \cdots$. To find an extremum for I, the expression is differentiated successively with respect to each C and each resulting

equation is set equal to zero. This provides r simultaneous equations for the unknown constants $C_1, \cdots, C_r$. In practice, this procedure may be applied successively using an increasing number of terms in the series (14) until convergence to a desired accuracy is achieved. The choice of functions $f_r(X)$ may be arbitrary, but better results (for a given number of terms) and more rapid convergence are achieved when the selections are made in accordance with any physical insight or knowledge which one may have about the problem (e.g., symmetry).

To fix these ideas about the variational procedure, application will be made to the parallel plate system which has previously been discussed. It is readily seen that the governing equation (6) for that problem is actually a special case of the general form (12) to which the variational method applies. As a matter of fact, if variable (but identical) surface temperatures and a radiating environment had been included in the parallel plate problem, the variational method would still be usable. These added effects would contribute nonintegral terms containing functions of X to the right side of equation (6); and these would correspond to G(X) in equation (12).

APPLICATION TO THE PARALLEL PLATE SYSTEM

As has already been noted, the parallel plate system is ideally suited for illustrating the way in which the variational procedure is used in solving a class of integral equations which arise in radiation problems. The parallel plate problem is governed by the integral equation (6), and the first step in obtaining its variational solution is to find the variational expression I. Comparing equation (6) to the general form (12), it follows that

$$\lambda = \frac{\rho r^2}{2}, \qquad G(X) = 1, \qquad a = -\frac{1}{2}, \qquad b = \frac{1}{2}$$

$$K(Y,X) = \frac{1}{\left[(Y - X)^2 + r^2\right]^{3/2}}$$

Substituting these into equation (13), the variational expression corresponding to equation (6) takes the form

$$I = \frac{\rho\gamma^2}{2}\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \frac{\beta(X)\beta(Y)\,dXdY}{[(Y - X)^2 + \gamma^2]^{3/2}} - \int_{-1/2}^{1/2} [\beta(x)]^2\,dX + 2\int_{-1/2}^{1/2} \beta(X)dX \tag{15}$$

To find an extremum of equation (15), we use the Ritz method as previously outlined. In selecting the functions $f_n(X)$ in equation (14), cognizance is taken of the fact that symmetry exists about the points $X = 0$ and $Y = 0$; that is, $\beta(X) = \beta(-X)$ and $\beta(Y) = \beta(-Y)$. Functions which immediately suggest themselves as candidates for the role of f_n are $X^0, X^2, X^4, \cdots$. Other possibilities exist (e.g., the cosines), but none appear quite as attractive as these power functions. So, as a first approximation for β in the form of equation (14), we try

$$\beta(X) = C_1 + C_2X^2, \qquad \beta(Y) = C_1 + C_2Y^2 \tag{16}$$

The constants C_1 and C_2 are determined so as to provide an extremum for I. Introducing these expressions for β into the variational expression (15) and carrying out the indicated integrations leads to

$$I = \rho(C_1^2a_1 + C_1C_2a_2 + C_2^2a_3) - C_1^2 - \frac{1}{6}C_1C_2 - \frac{1}{80}C_2^2 + 2C_1 + \frac{C_2}{6} \tag{17}$$

where the a_1, a_2, and a_3 depend only on the gap ratio γ and are given in appendix A. To find the values of C_1 and C_2 which give an extremum for I, we differentiate equation (17) with respect to each C and set the resulting equations to zero. Carrying out this operation, there is obtained

$$\left.\begin{aligned} \frac{\partial I}{\partial C_1} = 0 &\longrightarrow 2C_1(\rho a_1 - 1) + C_2\left(\rho a_2 - \frac{1}{6}\right) = -2 \\ \frac{\partial I}{\partial C_2} = 0 &\longrightarrow C_1\left(\rho a_2 - \frac{1}{6}\right) + 2C_2\left(\rho a_3 - \frac{1}{80}\right) = -\frac{1}{6} \end{aligned}\right\} \tag{18}$$

Simultaneous solution of this simple pair of linear algebraic equations gives C_1 and C_2. Once the C values have been found, the local and over-all heat transfer are easily computed by using equation (7a) and (8), giving

$$\frac{q}{\epsilon\sigma T^4} = \frac{1 - \epsilon(C_1 + C_2X^2)}{\rho} \tag{19}$$

$$\frac{Q}{\epsilon\sigma T^4} = \frac{1 - \epsilon(C_1 + C_2/12)}{\rho} \tag{20}$$

As a specific numerical example, consider the case of $\gamma = 0.5$. Then, evaluating equations (A1), (A2), and (A3), there is obtained: $a_1 = 0.61803$, $a_2 = 0.091237$, $a_3 = 0.0038127$. For $\rho = 0.9$, equation (18) becomes

$$0.88755\ C_1 + 0.084554\ C_2 = 2$$

$$0.084554\ C_1 + 0.018137\ C_2 = 1/6$$

from which it is found that $C_1 = 2.4789$ and $C_2 = -2.3673$. With these, the results for $B(X)$, $q(X)$, and Q follow from equations (16), (19), and (20), as

$$\beta = \frac{B}{\epsilon\sigma T^4} = 2.4789 - 2.3673\ X^2$$

$$\frac{q}{\epsilon\sigma T^4} = 0.8357 + 0.2630\ X^2, \quad \frac{Q}{\epsilon\sigma T^4} = 0.85759$$

where we have used $\epsilon = 1 - \rho = 0.1$. These results illustrate two characteristics which are generally true for the parallel plate problem regardless of the specific values of γ and ρ. First, the combined radiant flux B (emitted plus reflected) is a maximum at the mid-point of the plate ($X = 0$) and decreases steadily to a minimum value at the ends of the gap ($X = \pm 1/2$). Secondly, the local heat transfer q is smallest at the mid section and largest at the ends of the gap.

Values of C_1 and C_2 have been evaluated from equation (18) for gap spacings γ of 1, 0.5, and 0.1 for a wide range of reflectivities. The C values thus obtained are presented in table BI, appendix B.

The quadratic form for $\beta(X)$, as given by equation (16), was selected as a first approximation. It is desirable to know how well the resulting predictions compare with those of a more exact solution. Fortunately, there are available (ref. 4) some solutions of equation (6) which were obtained by direct iteration on an electronic computer. Using these as a standard, we proceed to test the accuracy of the quadratic variational solution. It may be first observed that for a given spacing γ, the variation of the combined flux β along the plate is

greater as ρ increases. Consequently, for a given γ, it was found that the most severe test of the variational procedure is at the larger ρ values. For this reason, detailed comparisons will only be presented for $\rho = 0.9$. Such comparisons are made in tables I(a), (b), and (c) for γ values of 1, 0.5, and 0.1, respectively at X values of 0, 0.1, 0.2, 0.3, 0.4, and 0.5. Attention in direction to the first and last lines of each table. The accuracy of the variational quadratic is seen to be remarkably good for $\gamma = 1$. As the gap ratio γ decreases, the predictions of the quadratic become somewhat less precise; but still are quite good. For the case of $\gamma = 0.1$, the errors range from 2 percent at $X = 0$ (center point) to 9 percent at $X = \pm 1/2$ (end points), with even smaller errors at some in-between points. The performance of the variational quadratic must be regarded as particularly gratifying, especially in view of its simplicity and ease of application.

In practice, the next step would be to carry out a second variational computation using a more complex form for β. From this, there would be obtained a more accurate set of radiation predictions. Further, in the absence of an exact solution, more than one variational computation is needed in order to define the accuracy of the results.

To demonstrate how the performance of the variational method can be improved, the procedure is repeated with the following quartic representation for β.

$$\beta(X) = D_1 + D_2X^2 + D_3X^4, \qquad \beta(Y) = D_1 + D_2Y^2 + D_3Y^4 \tag{21}$$

This form is a logical refinement of equation (16) and continues to incorporate the symmetry properties of β. Introducing equation (21) into the variational expression I gives

$$\begin{aligned} I = \rho(&D_1^2a_1 + D_1D_2a_2 + D_2^2a_3 + D_1D_2a_4 + D_2D_3a_5 + D_3^2a_6) \\ &- D_1^2 - \frac{1}{6}D_1D_2 - \frac{1}{40}D_1D_3 - \frac{D_2^2}{80} - \frac{D_2D_3}{224} \\ &- \frac{D_3^2}{2304} + 2D_1 + \frac{D_2}{6} + \frac{D_3}{40} \end{aligned} \tag{22}$$

where again, the $a_1, \cdots, a_6$ depend only on γ and are given in appendix A. To find the extremum of I, equation (22) is differentiated successively with respect to each D and the resulting equations set to zero. So,

$$\frac{\partial I}{\partial D_1} = 0 \longrightarrow 2D_1(\rho a_1 - 1) + D_2\left(\rho a_2 - \frac{1}{6}\right) + D_3\left(\rho a_4 - \frac{1}{40}\right) = -2$$

$$\frac{\partial I}{\partial D_2} = 0 \longrightarrow D_1\left(\rho a_2 - \frac{1}{6}\right) + 2D_2\left(\rho a_3 - \frac{1}{80}\right) + D_3\left(\rho a_5 - \frac{1}{224}\right) = -\frac{1}{6} \qquad (23)$$

$$\frac{\partial I}{\partial D_3} = 0 \longrightarrow D_1\left(\rho a_4 - \frac{1}{40}\right) + D_2\left(\rho a_5 - \frac{1}{224}\right) + 2D_3\left(\rho a_6 - \frac{1}{2304}\right) = -\frac{1}{40}$$

Once γ (i.e., a_1 through a_6) and ρ have been prescribed, then these equations can be readily solved for D_1, D_2, and D_3. With these, the local and over-all heat transfer results take the form

$$\frac{q}{\epsilon\sigma T^4} = \frac{1 - \epsilon(D_1 + D_2X^2 + D_3X^4)}{\rho} \qquad (24)$$

$$\frac{Q}{\epsilon\sigma T^4} = \frac{1 - \epsilon(D_1 + D_2/12 + D_3/80)}{\rho} \qquad (25)$$

Solutions of equations (23) have been carried out for γ values of 0.5 and 0.1* for a wide range of ρ values. The D_1, D_2, and D_3 thus computed are given in table BII, appendix B. Using these values, we can now proceed to compare predictions of the variational quartic with those of the quadratic and of the exact solution. Turning to tables I(b) and (c), it is observed that indeed, the quartic is closer to the exact solution than is the quadratic. For example, at $X = 0.5$ for $\gamma = 0.1$, the prediction as given by the quadratic is in error by 9 percent; while the quartic has cut the error to 0.6 percent.

Even closer agreement with the exact solution would be achieved if a sixth degree polynomial were to be used in place of the quartic. With an exact solution available for comparison purposes, it is clear that such a step is quite unnecessary for γ values of 0.1 and greater. However, in the absence of an exact solution, it might be necessary (depending on the precision desired) to use the sixth degree equation to verify the accuracy of the variational solution.

*No solutions were obtained for $\gamma = 1$, since the quadratic approximation has already provided exceedingly good results for that situation.

CONCLUDING REMARKS: LIMITATIONS

It has been demonstrated that a variational method can be used with considerable advantage to solve integral equations which may arise in thermal radiation problems. The important restriction which must be met is that the governing integral equation of the radiation problem have the form of equation (12). The note of caution to be sounded here is that the $\beta(Y)$ which appears under the integral must be the same function as the $\beta(X)$ which appears on the left side of the equation.

REFERENCES

1. Eckert, E. R. G., "Heat and Mass Transfer," McGraw-Hill Book Co., Inc., New York, N. Y., 1959.

2. Jakob, Max, "Heat Transfer," vol. II, John Wiley and Sons, Inc., New York, N. Y., 1957.

3. Hildebrand, F. B., "Methods of Applied Mathematics," Prentice-Hall, Inc., New York, N. Y., 1952.

4. Gregg, J. L., personal communication

TABLE I. - COMPARISON OF $\beta(x)$ FROM VARIOUS COMPUTATIONS

(a) $\gamma = 1$, $\rho = 0.9$.

X →	0	0.1	0.2	0.3	0.4	0.5
Quadratic	1.642	1.637	1.620	1.592	1.554	1.504
Exact	1.644	1.638	1.620	1.590	1.552	1.508

(b) $\gamma = 0.5$, $\rho = 0.9$.

X →	0	0.1	0.2	0.3	0.4	0.5
Quadratic	2.479	2.455	2.384	2.266	2.100	1.887
Quartic	2.486	2.460	2.382	2.258	2.096	1.905
Exact	2.485	2.459	2.383	2.259	2.095	1.908

(c) $\gamma = 0.1$, $\rho = 0.9$.

X →	0	0.1	0.2	0.3	0.4	0.5
Quadratic	7.39	7.22	6.73	5.90	4.74	3.25
Quartic	7.21	7.10	6.76	6.07	4.88	2.95
Exact	7.22	7.10	6.75	6.07	4.90	2.97

APPENDIX A: EXPRESSIONS FOR $a_1, \ldots, a_6$

$$a_1 = \sqrt{1 + r^2} - r \tag{A1}$$

$$a_2 = \left(\frac{1}{6} - \frac{4r^2}{3}\right)\sqrt{1 + r^2} + r^2\left[\ln(1 + \sqrt{1 + r^2}) - \ln r\right] + \frac{4r^3}{3} - \frac{r}{2} \tag{A2}$$

$$a_3 = \left(\frac{1}{80} - \frac{2r^2}{45} + \frac{4r^4}{45}\right)\sqrt{1 + r^2} + \frac{r^2}{12}\left[\ln(1 + \sqrt{1 + r^2}) - \ln r\right] - \frac{r}{16} - \frac{4r^5}{45} \tag{A3}$$

$$a_4 = \left(\frac{1}{40} - \frac{31r^2}{30} + \frac{16r^4}{15}\right)\sqrt{1 + r^2} + \left(\frac{r^2}{2} - \frac{3r^4}{2}\right)\left[\ln(1 + \sqrt{1 + r^2}) - \ln r\right]$$
$$- \frac{r}{8} + 2r^3 - \frac{16r^5}{15} \tag{A4}$$

$$\frac{a_5}{2} = \left(\frac{1}{448} - \frac{726r^2}{8400} + \frac{4468r^4}{8400} - \frac{296r^6}{525}\right)\sqrt{1 + r^2} + \left(\frac{7r^2}{160} - \frac{5r^4}{16} + \frac{3r^6}{4}\right) \cdot$$
$$\left[\ln(1 + \sqrt{1 + r^2}) - \ln r\right] - \frac{r}{64} + \frac{3r^3}{16} - r^5 + \frac{296r^7}{525} + \frac{r^2}{2} a_4 \tag{A5}$$

$$a_6 = \int_{-1/2}^{1/2} \xi^4 \, g(\xi) d\xi \tag{A6}$$

$$g(\xi) = \sqrt{(1/2 - \xi)^2 + r^2}\left[\frac{1}{2}\left(\frac{1}{2} - \xi\right) + 4\xi\right]$$
$$+ \left[\left(\frac{1}{2} - \xi\right)^2 + r^2\right]^{-1/2}\left[\left(\frac{1}{2} - \xi\right)\left(r^2 - 6\xi^2 + \frac{\xi^4}{r^2}\right) + 4\xi(r^2 - \xi^2)\right]$$
$$+ \sqrt{\left(\frac{1}{2} + \xi\right)^2 + r^2}\left[\frac{1}{2}\left(\frac{1}{2} + \xi\right) - 4\xi\right]$$
$$+ \left[\left(\frac{1}{2} + \xi\right)^2 + r^2\right]^{-1/2}\left[\left(\frac{1}{2} + \xi\right)\left(r^2 - 6\xi^2 + \frac{\xi^4}{r^2}\right) - 4\xi(r^2 - \xi^2)\right]$$
$$+ \left(6\xi^2 - \frac{3r^2}{2}\right)\ln\left[\frac{\left(\frac{1}{2} - \xi\right) + \sqrt{\left(\frac{1}{2} - \xi\right)^2 + r^2}}{-\left(\frac{1}{2} + \xi\right) + \sqrt{\left(\frac{1}{2} + \xi\right)^2 + r^2}}\right]$$

With respect to the a_6 computation, it was found expedient to evaluate the integral numerically (by Simpson's rule) rather than analytically.

APPENDIX B: NUMERICAL VALUES OF C_n AND D_n

TABLE BI

Values of C_1 and C_2 for Quadratic, eq. (16)

	$\gamma = 1$		$\gamma = 0.5$		$\gamma = 0.1$	
ϵ	C_1	$-C_2$	C_1	$-C_2$	C_1	$-C_2$
0.1	1.642	0.5549	2.479	2.367	7.388	16.54
.3	1.441	.3801	1.890	1.403	3.192	4.726
.5	1.282	.2423	1.518	.8045	1.992	1.832
.7	1.153	.1315	1.262	.4013	1.433	.6996
.9	1.047	.03998	1.075	.1139	1.113	.1624

TABLE BII

Values of D_1, D_2, and D_3 for Quartic, eq. (21)

	$\gamma = 0.5$			$\gamma = 0.1$		
ϵ	D_1	$-D_2$	D_3	D_1	$-D_2$	$-D_3$
0.1	2.486	2.644	1.284	7.205	10.06	27.81
.3	1.893	1.548	.6726	3.095	1.128	15.92
.5	1.519	.8754	.3306	1.942	-.06314	8.564
.7	1.263	.4322	.1440	1.411	-.1774	4.024
.9	1.076	.1210	.03261	1.107	-.06997	1.079

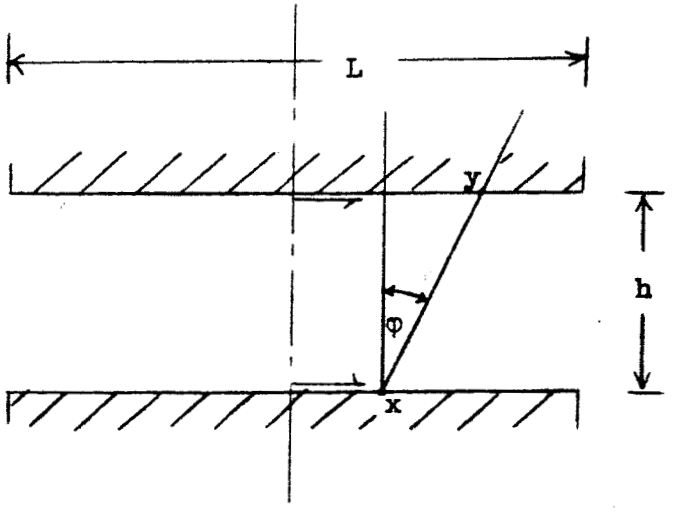

Figure 1. - Schematic of parallel plate system.